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ISOMETRIES BETWEEN BANACH SPACES OF LIPSCHITZ FUNCTIONS†

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ABSTRACT

The Banach spaces $\mathrm{Lip}^\alpha(S,\Delta)$, $\mathrm{lip}^\alpha(S,\Delta)$, $\mathrm{Lip}^\alpha(S,\Delta;s_0)$ and $\mathrm{lip}^\alpha(S,\Delta;s_0)$ of Lipschitz functions are defined. We shall identify the extreme points of the unit balls in their corresponding dual spaces and make use of them to present a complete characterization of the isometries between these function spaces.

1. Introduction and definitions

Throughout this paper the following notation will be adopted. If E is a Banach space, B_E is its unit ball and E' its dual space, and if $A \subset E$ then $\mathrm{ext}(A)$ is the set of all the extreme points in A. We shall deal with real functions defined on a compact metric space (S,Δ). Such a function f satisfies a Lipschitz condition of order α, $0<\alpha\leqq 1$, if

$$\|f\|_\alpha = \sup_{s\neq t} \frac{|f(s)-f(t)|}{\Delta^\alpha(s,t)} < \infty. \tag{1.1}$$

We define:

$$\mathrm{Lip}^\alpha(S,\Delta) = \{f : S \to R / f \text{ satisfies } (1.1)\}.$$

Endowed with the norm $\|f\| = \max(\|f\|_\alpha, \|f\|_\infty)$, $\mathrm{Lip}^\alpha(S,\Delta)$ is a Banach space.

Of special interest are the functions f which satisfy

$$\lim_{\delta\to 0} \sup_{0<\Delta^\alpha(s,t)<\delta} \frac{|f(s)-f(t)|}{\Delta^\alpha(s,t)} = 0. \tag{1.2}$$

Namely, we define

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$$\mathrm{lip}^\alpha(S,\Delta)=\{f\in \mathrm{Lip}^\alpha(S,\Delta)/f \text{ satisfies } (1.2)\}.$$

It is easy to check that $\mathrm{lip}^\alpha(S,\Delta)$ is a closed separable subspace of $\mathrm{Lip}^\alpha(S,\Delta)$.

Alternatively one can choose an element $s_0\in S$ and define:

$$\mathrm{Lip}^\alpha(S,\Delta;s_0)=\{f:S\to R\,/\,f(s_0)=0 \text{ and } f \text{ satisfies } (1.2)\}$$

and

$$\mathrm{lip}^\alpha(S,\Delta;s_0)=\{f\in \mathrm{Lip}^\alpha(S,\Delta;s_0)/f \text{ satisfies } (1.2)\}.$$

Endowed with the norm $\|f\|_\alpha$, $\mathrm{Lip}^\alpha(S,\Delta;s_0)$ becomes a Banach space of which $\mathrm{lip}^\alpha(S,\Delta;s_0)$ is a closed separable subspace. It is not hard to see that if card $S=\infty$ then $\mathrm{Lip}^\alpha(S,\Delta)$ and $\mathrm{Lip}^\alpha(S,\Delta;s_0)$ are isomorphic. ($\mathrm{Lip}^\alpha(S,\Delta;s_0)$ is clearly isomorphic to a hyperplane of $\mathrm{Lip}^\alpha(S,\Delta)$, and the latter is isomorphic to all of its hyperplanes since it contains a subspace isomorphic to l_∞ ([5, theorem 1]).) Similarly it follows from [5] that if $\alpha<1$, $\mathrm{lip}^\alpha(S,\Delta)$ contains a subspace isomorphic to c_0 and is therefore isomorphic to $\mathrm{lip}^\alpha(S,\Delta;s_0)$.

If X is one of the four spaces defined above, for every $s\in S$, $\delta_s\in X'$ will denote the evaluation functional, and for every pair of distinct elements (s,t) in S, $\delta_{s,t}\in X'$ will be defined by

$$\delta_{s,t}(f)=\frac{f(s)-f(t)}{\Delta^\alpha(s,t)}\qquad \forall f\in X.$$

Whenever reference is made to one of these functionals, it will be clear from the context which space X will be.

The isomorphic classification of the Banach spaces of Lipschitz functions has not yet been solved. It is conjectured that for $0<\alpha<1$, $\mathrm{Lip}^\alpha(S,\Delta)$ is isomorphic to l_∞ and $\mathrm{lip}^\alpha(S,\Delta)$ is isomorphic to c_0. This has been proved for S a compact set in R^n (see [1]). On the other hand, it has been observed by Y. Benyamini and P. Wojtaszczyk that it follows from J. Kislyakov's paper [7] that $\mathrm{Lip}^1(I^2,d)$ is not isomorphic to l_∞ (I^2 being the unit square in R^2 with its natural metric d). Another counterexample for $\alpha=1$ can be found in [6].

In the present paper we deal with the isometric classification. We characterize the isometries between pairs of spaces of the same type, for each of the four types defined above (for $\alpha<1$). Some particular cases have already been studied. In [2] the metric spaces were taken to be circles of unit circumference in R^2 and it was shown that all isometries $R:\mathrm{lip}^\alpha(H,d)\to \mathrm{lip}^\alpha(K,\rho)$ are given by a composition with a distance preserving map $\varphi:K\to H$, i.e.

$$Tf(x)=\theta f(\varphi(x))\qquad \forall f\in \mathrm{lip}^\alpha(H,d)$$

where $\theta = \pm 1$. The same conclusion was obtained in [9] where the metric spaces were general Riemannian manifolds with the Riemannian metric.

In Section 3 we generalize these results by considering general compact metric spaces. We explicitly point out certain isometries which are not "composition isometries", as mentioned above. (In this case the involved metric spaces possess a very particular structure; we call such spaces 1-centered metric spaces.) We then show that all the isometries are actually generated, in a certain sense, by the "composition isometries" and this second type of isometries.

We also study the isometries $T : \mathrm{lip}^\alpha(H, d; x_0) \to \mathrm{lip}^\alpha(K, \rho; u_0)$ which are somewhat simpler than those in the previous case. We then proceed to characterize all the isometries

$$T : \mathrm{Lip}^\alpha(H, d) \to \mathrm{Lip}^\alpha(K, \rho) \quad \text{and} \quad T : \mathrm{Lip}^\alpha(H, d; x_0) \to \mathrm{Lip}^\alpha(K, \rho; u_0)$$

and show that they have essentially the same form as those between the corresponding subspaces.

This last fact follows from a topological property of the extreme points of the respective dual unit balls which is developed in Section 2. On the whole, these extreme points play a key role in our characterization of the isometries and Section 2 is devoted to identify them in each case.

We have allowed ourselves two simplifications. In the first place only real functions are considered. It may be easily verified that for the corresponding complex spaces our results remain essentially the same (the only changes needed usually consist of replacing "$\theta = \pm 1$" by "$|\theta| = 1$").

Secondly, our classification includes only the case in which the domain and the range of the isometries consist of functions which satisfy a Lipschitz condition of the same order. This is justified by the fact that if $\beta < \alpha$, the functions defined on a metric space (S, Δ) which satisfy a Lipschitz condition of order β can be looked upon as the functions which satisfy a Lipschitz condition of order α on $(S, \Delta^{\beta/\alpha})$.

For further general facts about Lipschitz function spaces the reader is referred to [4].

2. Extreme points of the unit balls of the dual spaces

We state without proof a lemma from [2] and [3] which will serve as a basis to identify the extreme points of the unit dual balls of the four spaces defined in the introduction.

LEMMA 2.1. *Let K be a compact Hausdorff space and $A \subset C(K)$ a closed subspace. Then*

(a) [3, V 8.6] *Every extreme point of $B_{A'}$ can be extended to an element of $C(K)'$ of the form $\theta\delta_y$ with $y \in K$ and $\theta = \pm 1$.*

(b) [2, lemma 3.2] *If $y \in K$, a sufficient condition for $\delta_y \mid_A$ to be an extreme point of $B_{A'}$ is that there exists a function $f \in B_A$ such that*

(i) $f(y) = 1$,

(ii) $|f(x)| = 1$ *iff there exists a $\theta = \pm 1$ such that*

$$g(x) = \theta g(y) \qquad \forall g \in A.$$

In this case we say that f peaks at y relative to A.

REMARK. Lemma 2.1 clearly remains true if we replace $C(K)$ by $C_0(Y)$, with Y a locally compact Hausdorff space.

The following theorem essentially generalizes [2, theorem 3.3].

THEOREM 2.1. *Let (S, Δ) be a compact metric space, $0 < \alpha < 1$, and $s_0 \in S$. Then*

(a) $\operatorname{ext}(B_{\operatorname{lip}^\alpha(S,\Delta)'}) = \{\pm \delta_s / s \in S\} \cup \{\delta_{s,t} \in S,\ 0 < \Delta^\alpha(s,t) < 2\}$,

(b) $\operatorname{ext}(B_{\operatorname{lip}^\alpha(S,\Delta;s_0)'}) = \{\delta_{s,t} / s, t \in S, s \neq t\}$.

PROOF. We first point out that if we define

$$W = S \times S \setminus \operatorname{diag}(S \times S) \quad \text{and} \quad Z = S \cup W \quad \text{(disjoint union)}$$

then the map $r_1 : \operatorname{lip}^\alpha(S, \Delta) \to C_0(Z)$ defined by

$$r_1 f(z) \equiv \begin{cases} f(s) & z = s \in S \\ \dfrac{f(s) - f(t)}{\Delta^\alpha(s,t)} & z = (s,t) \in W \end{cases}$$

imbeds $\operatorname{lip}^\alpha(S, \Delta)$ isometrically as a subspace of $C_0(Z)$. Similarly the map $r_2 : \operatorname{lip}^\alpha(S, \Delta; s_0) \to C_0(W)$ defined by

$$r_2 f(s,t) = \frac{f(s) - f(t)}{\Delta^\alpha(s,t)}, \qquad (s,t) \in W$$

imbeds $\operatorname{lip}^\alpha(S, \Delta; s_0)$ isometrically as a subspace of $C_0(W)$.

We now proceed with the proof of (a). It follows from Lemma 2.1 and the remark following it that every extreme point of $B_{\operatorname{lip}^\alpha(S,\Delta)'}$ is of the form $\pm\delta_s$,

$s \in S$, or $\delta_{s,t}$, $(s,t) \in W$. On the other hand the equality $\delta_{s,t} = \Delta^{-\alpha}(s,t)(\delta_s - \delta_t)$ shows that whenever $\Delta^\alpha(s,t) \geqq 2$, $\delta_{s,t} \notin \operatorname{ext}(B_{\mathrm{lip}^\alpha(S,\Delta)'})$.

According to Lemma 2.1 (b) it remains to be shown that for each remaining $z \in Z$ there exists a function $f_z \in \mathrm{lip}^\alpha(S,\Delta)$ such that $r_1 f_z$ peaks at z relative to $\mathrm{lip}^\alpha(S,\Delta)$. For every $s \in S$ we set $f_s(x) = 1 - \lambda\Delta(s,x)$. It can be easily verified that by choosing λ sufficiently small $(\lambda < \min(1, \mathrm{diam}^{\alpha-1}(S)))$ $r_1 f_s$ peaks at s relative to $r_1(\mathrm{lip}^\alpha(S,\Delta))$.

If $z = (s,t) \in W$, $\Delta^\alpha(s,t) < 2$, we choose a real number β, $\alpha < \beta < 1$, and set

$$f_{(s,t)}(x) = \frac{\Delta^\beta(t,x) - \Delta^\beta(s,x)}{2\Delta^{\beta-\alpha}(s,t)}. \tag{2.1}$$

Clearly $f_{(s,t)} \in \mathrm{lip}^\alpha(S,\Delta)$, $r_1 f_{(s,t)}(s,t) = 1$ and

$$|r_1 f_{(s,t)}(x)| = |f_{(s,t)}(x)| \leqq \frac{\Delta^\alpha(s,t)}{2} < 1 \qquad \forall x \in S.$$

Finally, if $w = (s',t') \in W$ then

$$|r_1 f_{(s,t)}(w)| = \frac{|f_{(s,t)}(s') - f_{(s,t)}(t')|}{\Delta^\alpha(s',t')} = \frac{|\Delta^\beta(s,s') - \Delta^\beta(t,s') + \Delta^\beta(t,t') - \Delta^\beta(s,t')|}{2\Delta^{\beta-\alpha}(s,t)\Delta^\alpha(s',t')}$$

$$\leqq \frac{2\min(\Delta^\beta(s',t'), \Delta^\beta(s,t))}{2\Delta^{\beta-\alpha}(s,t)\Delta^\alpha(s',t')} = \min\left(\left(\frac{\Delta(s,t)}{\Delta(s',t')}\right)^\alpha, \left(\frac{\Delta(s',t')}{\Delta(s,t)}\right)^{\beta-\alpha}\right) \leqq 1.$$

Therefore

$$|r_1 f_{(s,t)}(w)| \leqq 1. \tag{2.2}$$

Since $\beta < 1$, equality occurs in the triangle inequality for Δ^β if and only if at least two of the three points involved coincide. In our case this means that equality holds in (2.2) if and only if $w = (s,t)$ or $w = (t,s)$. According to the definition, $r_1 f_{(s,t)}$ peaks at (s,t) relative to $r_1(\mathrm{lip}^\alpha(S,\Delta))$.

The proof of (b) is very much the same. In view of the imbedding r_2, it follows from Lemma 2.1 that we only have to find for each $(s,t) \in W$ a function $g_{(s,t)}$ which peaks at (s,t) relative to $r_2(\mathrm{lip}^\alpha(S,\Delta;s_0))$. To that effect let us define

$$\tilde{g}_{(s,t)}(x) = \frac{\Delta^\beta(t,x) - \Delta^\beta(s,x)}{2\Delta^{\beta-\alpha}(s,t)}$$

as in (2.1), $(\alpha < \beta < 1)$, and $g_{(s,t)}(x) = \tilde{g}_{(s,t)}(x) - \tilde{g}_{(s,t)}(s_0)$.

The fact that $g_{(s,t)}$ fulfils the requirements follows exactly as in the proof of part (a). □

REMARK. It is a consequence of Theorem 2.1 (and also easy to verify directly) that to calculate the norm of a function in $\mathrm{lip}^\alpha(S,\Delta)$ (actually in $\mathrm{Lip}^\alpha(S,\Delta)$ as well) it is sufficient to consider the Lipschitz condition locally, i.e.

$$\|f\| = \max\left(\|f\|_\infty, \sup_{0<\Delta^\alpha(s,t)<2} \frac{|f(s)-f(t)|}{\Delta^\alpha(s,t)}\right).$$

We shall now deal with the extreme points of $B_{\mathrm{Lip}^\alpha(S,\Delta)'}$. To that effect we consider $\mathrm{Lip}^\alpha(S,\Delta)$ as a subspace of $C(\beta Z)$ (βZ is the Stone–Čech compactification of Z). The isometric imbedding Γ is given by

$$\Gamma f(z) = \begin{cases} f(s) & z = s \in S, \\ \dfrac{f(s)-f(t)}{\Delta^\alpha(s,t)} & z = (s,t) \in W. \end{cases}$$

(Γf is uniquely extended to βZ as a bounded continuous function.) According to Lemma 2.1 (a) the extreme points of $B_{\mathrm{Lip}^\alpha(S,\Delta)'}$ are of the form $\pm\delta_x$, $x \in \beta Z$. It is easy to see, as in Theorem 2.1, that for each $s \in S$, δ_s is an extreme point of $B_{\mathrm{Lip}^\alpha(S,\Delta)'}$ and if $(s,t) \in W$ then $\delta_{s,t}$ is an extreme point of $B_{\mathrm{Lip}^\alpha(S,\Delta)'}$ if and only if $\Delta^\alpha(s,t) < 2$.

It is not so simple to identify exactly

$$A = \{\delta_x, x \in \beta Z \setminus Z, \delta_x \in \mathrm{ext}(B_{\mathrm{Lip}^\alpha(S,\Delta)'})\}.$$

It has been observed by J. Johnson [4, theorem 2.8] that if S is of infinite cardinality then A is not empty. We shall just point out that $\beta Z \setminus Z = \beta W \setminus W$ and that for every $x \in \beta W \setminus W$, δ_x can be viewed as a generalized derivative functional at some point $s_x \in S$. Indeed, if (s_ν, t_ν) is a net in W converging to x, we can assume by passing to a subnet if necessary, that $s_\nu \to s$ and $t_\nu \to t$. If $s \neq t$ we have $x = (s,t) \in W$ contrary to the assumption. Thence $s = t = s_x$ (and s_x is clearly independent of the chosen converging net).

These functionals are actually called point derivatives (see [8]). In general, if $s \in S$ there are "many" point derivatives associated with s, according to how (s_ν, t_ν) tends to (s,s). Further results concerning point derivatives appear in [4] and [8]. For our purpose it will suffice to partially sum up these remarks in the following theorem. For the remainder of this section we will denote

$$F(S) = \{\pm\delta_s / s \in S\} \cup \{\delta_{s,t} / s,t \in S, 0 < \Delta^\alpha(s,t) < 2\} \subset \mathrm{Lip}^\alpha(S,\Delta)'.$$

THEOREM 2.2.

$$F(S) \subset \mathrm{ext}(B_{\mathrm{Lip}^\alpha(S,\Delta)'}) \subset \omega^*\text{-cl}\, F(S).$$ □

We shall now present a topological distinction between those extreme points which belong to $F(S)$ and those which don't.

THEOREM 2.3. *Suppose $\varphi \in \mathrm{ext}(B_{\mathrm{Lip}^\alpha(S,\Delta)'})$. Then φ has a ω^*-metrizable neighbourhood in ω^*-cl $F(S)$ if and only if $\varphi \in F(S)$.*

PROOF. Since $F(S)$ (in its ω^*-topology) and Z are locally homeomorphic, the proof of the "if" assertion is immediate.

To prove the other implication suppose $\varphi \not\in F(S)$. We shall show that the existence of a metrizable neighbourhood of φ leads to a contradiction. Indeed, if such a neighbourhood exists then there is a sequence $\{\varphi_n\}_{n=1}^\infty \subset F(S)$ such that $\varphi_n \to \varphi$ in the ω^* topology. In view of the comments leading to Theorem 2.2 we can assume without loss of generality that

$$\varphi_n = \delta_{s_n,t_n} \quad \text{with } s_n \to s \quad \text{and} \quad t_n \to s$$

(that is, φ is a point derivative at s). We shall arrive at a contradiction by constructing a function $f \in \mathrm{Lip}^\alpha(S,\Delta)$ such that $\lim_{n\to\infty} \delta_{s_n,t_n}(f)$ does not exist.

To that effect, for every $(s,t) \in W$, we consider a function $f_{s,t} \in \mathrm{lip}^\alpha(S,\Delta)$ which satisfies

(i) $\|f_{s,t}\|_\alpha = (f_{s,t}(s) - f_{s,t}(t))/\Delta^\alpha(s,t) = 1$,

(ii) $\|f_{s,t}\|_\infty \leqq \Delta^\alpha(s,t)$.

(For example, we can take $f_{s,t}$ as defined by (2.1).) We now construct an increasing sequence $\{n_k\}_{k=1}^\infty$ of natural numbers and a decreasing sequence $\{\varepsilon_k\}_{k=1}^\infty$ of positive real numbers in the following way: choose $n_1 = 1$ and having defined n_k, $k \geqq 1$ (and denoting $f_k = f_{s_{n_k},t_{n_k}}$), choose $\varepsilon_k > 0$ such that whenever $\Delta^\alpha(s,t) < \varepsilon_k$,

$$\frac{|f_k(s) - f_k(t)|}{\Delta^\alpha(s,t)} \leqq \frac{1}{2^{k+2}}.$$

Then define n_{k+1} subject to the condition

$$\Delta^\alpha(s_{n_{k+1}}, t_{n_{k+1}}) < \frac{\varepsilon_k^\alpha}{5}.$$

Using (i) and the definition of ε_k and ε_{k+1}, it is easy to verify that for every $k \geqq 1$

$$\varepsilon_{k+1}^\alpha \leqq \Delta^\alpha(s_{n_{k+1}}, t_{n_{k+1}}) \leqq \frac{\varepsilon_k^\alpha}{5} \leqq \frac{\Delta^\alpha(s_{n_k}, t_{n_k})}{5}. \tag{2.3}$$

Hence $\sum_{k=1}^\infty \|f_k\|_\infty \leqq \sum_{k=1}^\infty \Delta^\alpha(s_{n_k}, t_{n_k}) < \infty$ and therefore $f = \sum_{k=1}^\infty (-1)^k f_k$ is well defined in $C(S)$. We shall now check that $\|f\|_\alpha < \infty$.

To that effect fix s, t in S, $s \neq t$, and denote $\varepsilon_0 = \operatorname{diam}(S)$. Then, clearly

$$\varepsilon_j < \Delta(s,t) \leqq \varepsilon_{j-1} \tag{2.4}$$

for some $j \geqq 1$. We have

$$\frac{f(s)-f(t)}{\Delta^\alpha(s,t)} = \sum_{k=1}^{j-1} (-1)^k \frac{(f_k(s)-f_k(t))}{\Delta^\alpha(s,t)} + (-1)^j \frac{(f_j(s)-f_j(t))}{\Delta^\alpha(s,t)} + \sum_{k=j+1}^{\infty} (-1)^k \frac{(f_k(s)-f_k(t))}{\Delta^\alpha(s,t)}.$$

Evaluating these terms separately we obtain

$$\left| \sum_{k=1}^{j-1} (-1)^k \frac{f_k(s)-f_k(t)}{\Delta^\alpha(s,t)} \right| \leqq \sum_{k=1}^{j-1} \frac{|f_k(s)-f_k(t)|}{\Delta^\alpha(s,t)} \leqq \sum_{k=1}^{j-1} \frac{1}{2^{k+2}} \leqq \frac{1}{4}, \tag{2.5}$$

$$\left| (-1)^j \frac{(f_j(s)-f_j(t))}{\Delta^\alpha(s,t)} \right| \leqq \|f_j\|_\alpha = 1, \tag{2.6}$$

$$\left| \sum_{k=j+1}^{\infty} (-1)^k \frac{(f_k(s)-f_k(t))}{\Delta^\alpha(s,t)} \right| \leqq \frac{2}{\varepsilon_j^\alpha} \sum_{k=j+1}^{\infty} \|f_k\|_\infty \leqq \left(\frac{\varepsilon_{j+1}^\alpha}{\varepsilon_j^\alpha}\right) \sum_{k=0}^{\infty} \left(\frac{1}{5}\right) \leqq \frac{1}{2}. \tag{2.7}$$

Hence, $\|f\|_\alpha < \infty$. Finally we prove that $\lim_{k\to\infty} \delta_{s_{n_k}, t_{n_k}}(f)$ doesn't exist. Setting $s = s_{n_j}$ and $t = t_{n_j}$, (2.4) holds. Therefore, by (2.5) and (2.7), we have, for even j's

$$\frac{f(s_{n_j})-f(t_{n_j})}{\Delta^\alpha(s_{n_j}, t_{n_j})} \geqq \frac{f_j(s_{n_j})-f_j(t_{n_j})}{\Delta^\alpha(s_{n_j}, t_{n_j})} - \left(\sum_{\substack{k=1 \\ k\neq j}}^{\infty} \frac{|f_k(s_{n_j})-f_k(t_{n_j})|}{\Delta^\alpha(s_{n_j}, t_{n_j})} \right) \geqq \frac{1}{4}.$$

Similarly, for odd j's we obtain

$$\frac{f(s_{n_j})-f(t_{n_j})}{\Delta^\alpha(s_{n_j}, t_{n_j})} \leqq -\frac{1}{4}$$

and thus reach the desired contradiction. □

REMARK. Theorem 2.2 and Theorem 2.3 remain true if we replace $\operatorname{Lip}^\alpha(S,\Delta)$ by $\operatorname{Lip}^\alpha(S,\Delta; s_0)$ and correspondingly define

$$F(S) = \{\delta_{s,t} / (s,t) \in W\}.$$

3. The isometric classification

Throughout this section (H, d) and (K, ρ) will be compact metric spaces, $x_0 \in H$, $u_0 \in K$ and $0 < \alpha < 1$. Let us point out that by the term isometry we denote a surjective norm preserving linear operator. We shall first characterize the operators

$$(3.1)\qquad T:\mathrm{lip}^\alpha(H,d;x_0)\to \mathrm{lip}^\alpha(K,\rho;u_0)$$

which are isometries.

THEOREM 3.1. *An operator T as in* (3.1) *is an isometry if and only if there exists a $\theta=\pm 1$ and a one to one surjection $\varphi:K\to H$ satisfying*

$$(3.2)\qquad \rho(u,v)=Cd(\varphi(u),\varphi(v))\qquad \forall u,v\in K$$

with $C=\mathrm{diam}(K)/\mathrm{diam}(H)$, *such that*

$$(3.3)\quad Tg(u)=\theta C^\alpha(g(\varphi(u))-g(\varphi(u_0)))\qquad \forall u\in K,\quad \forall g\in \mathrm{lip}^\alpha(H,d;x_0).$$

PROOF. It is trivial to see that an operator given by (3.3) is an isometry, so we shall only prove the converse statement. Assume T is an isometry. It is a known fact that T^* maps $\mathrm{ext}(B_{\mathrm{lip}^\alpha(K,\rho,u_0)'})$ onto $\mathrm{ext}(B_{\mathrm{lip}^\alpha(H,d,x_0)'})$. In other words defining $W_K=K\times K\setminus\mathrm{diag}(K\times K)$, there exist maps $\gamma_j:W_K\to H$, $j=1,2$, such that $T^*\delta_{u,v}=\delta_{\gamma_1(u,v),\gamma_2(u,v)}$ for each $(u,v)\in W_K$ (see Theorem 2.1(b)). Since clearly $\gamma_1(u,v)=\gamma_2(v,u)$ we conclude the existence of a map $\gamma:W_K\to H$ such that $T^*\delta_{u,v}=\delta_{\gamma(u,v),\gamma(v,u)}$, $(u,v)\in W_K$. Take now three different elements u, v, w of K. Since

$$\delta_{u,v}=\frac{\rho^\alpha(u,w)\delta_{u,w}-\rho^\alpha(v,w)\delta_{v,w}}{\rho^\alpha(u,v)}$$

we have

$$(3.4)\quad T^*\delta_{u,v}=\frac{\dfrac{\rho^\alpha(u,w)}{d^\alpha(\gamma(u,w),\gamma(w,u))}(\delta_{\gamma(u,w)}-\delta_{\gamma(w,u)})-\dfrac{\rho^\alpha(v,w)}{d^\alpha(\gamma(v,w),(w,v))}(\delta_{\gamma(v,w)}-\delta_{\gamma(w,v)})}{\rho^\alpha(u,v)}$$

Now, $T^*\delta_{u,v}$ is supported by at most two points in H, and since the δ_x's are linearly independent functionals two of the terms must cancel, and thus

$$(3.5)\qquad \gamma(u,w)=\gamma(v,w)$$

or

$$(3.5)'\qquad \gamma(w,u)=\gamma(w,v).$$

The next (purely combinatoric) lemma establishes that (3.5) always holds or (3.5)′ always holds ("always" meaning for every choice of distinct elements u, v, $w\in K$).

LEMMA 3.1. *Let S and U be 2 sets containing at least three distinct members, and $\gamma : S \times S \setminus \operatorname{diag}(S \times S) \to U$ a map which satisfies $\gamma(s, s') \neq \gamma(s', s)$ for all $s \neq s'$ such that the map $\Gamma : S \times S \setminus \operatorname{diag}(S \times S) \to U \times U \setminus \operatorname{diag}(U \times U)$ defined by $\Gamma(s, s') = (\gamma(s, s'), \gamma(s', s))$ is one to one. Assume that for every three distinct elements of S, s_1, s_2 and s_3,*

$$\gamma(s_2, s_1) = \gamma(s_3, s_1) \tag{3.6}$$

or

$$\gamma(s_1, s_2) = \gamma(s_1, s_3). \tag{3.6$'$}$$

Then either (3.6) *holds for all distinct* (s_1, s_2, s_3) *or* (3.6′) *holds for all distinct* (s_1, s_2, s_3).

We shall now proceed with the theorem's proof, assuming for the time being that the lemma has been established. Clearly our map satisfies the lemma's conditions by which we can conclude that $\gamma(u, v)$ is a function of only one of its arguments.

Suppose that $\gamma(u, v) = \varphi(u)$. It follows that

$$T^* \delta_{u,v} = \delta_{\gamma(u,v),\gamma(v,u)} = \delta_{\varphi(u),\varphi(v)} \tag{3.7}$$

and to get the cancellation in (3.4), we must have

$$\frac{\rho(u, w)}{d(\varphi(u), \varphi(w))} = \frac{\rho(v, w)}{d(\varphi(v), \varphi(w))}.$$

In other words, there exists a $C > 0$ such that (3.2) is satisfied and clearly $C = \operatorname{diam}(K)/\operatorname{diam}(H)$.

Finally, substituting u_0 for v in (3.7) we obtain (3.3) with $\theta = 1$. Similarly, had we assumed $\gamma(u, v) = \varphi(v)$ we would have arrived at (3.3) with $\theta = -1$. □

PROOF OF LEMMA 3.1. First we shall show that if (3.6) holds for some triple (s_1, s_2, s_3), then $\gamma(s, s_1) = \gamma(s_2, s_1)$ for all $s \in S$ (with the equivalent assertion holding if (3.6′) is true for some triple).

Assume, to the contrary, that there exists an $s \in S$ such that $\gamma(s_1, s) = \gamma(s_1, s_2)$. There are two possibilities:

(i) $\gamma(s_1, s) = \gamma(s_1, s_3)$ and thus $\gamma(s_1, s_2) = \gamma(s_1, s_3)$ which together with (3.6) implies that $\Gamma(s_1, s_2) = \Gamma(s_1, s_3)$, contradicting the assumption that Γ is one to one.

(ii) $\gamma(s, s_1) = \gamma(s_3, s_1)$. From (3.6) it then follows that $\gamma(s, s_1) = \gamma(s_2, s_1)$, a contradiction again.

Thus, for each $s \in S$, either

(a) $\gamma(s', s) = \gamma(s'', s)$ for all s', s'' in S, or

(b) $\gamma(s, s') = \gamma(s, s'')$ for all s', s'' in S.

We want to show that either (a) holds for every s or (b) holds for every s. Assume this is not so, that is, choose s_1 and s_2 such that for all s' and s'' in S, $\gamma(s', s_1) = \gamma(s'', s_1)$ and $\gamma(s_2, s') = \gamma(s_2, s'')$, and fix $s_3 \in S$ $(s_3 \neq s_1, s_2)$. Then

$$\gamma(s_3, s_1) = \gamma(s_2, s_1) = \gamma(s_2, s_3). \tag{3.8}$$

Now, if (a) holds (for $s = s_3$) then (3.8) yields $\gamma(s_3, s_1) = \gamma(s_2, s_3) = \gamma(s_1, s_3)$. Similarly if (b) holds we obtain from (3.8) $\gamma(s_3, s_2) = \gamma(s_3, s_1) = \gamma(s_2, s_3)$.

In both cases a contradiction to the lemma's assumption is reached. □

Let us now consider the isometry characterization for operators

$$T : \mathrm{lip}^\alpha (H, d) \to \mathrm{lip}^\alpha (K, \rho). \tag{3.9}$$

In this case, not all isometries are of the form (compare with (3.3))

$$Tf(u) = \theta C f(\varphi(u)), \qquad \theta = \pm 1, \quad C > 0. \tag{3.10}$$

Consider the following example: let $H = K = \{z \in \mathbf{C} / |z| = 1\} \cup \{0\}$ and d and ρ the natural metric. Choose $\theta_0 \in [0, 2\pi)$ and define an operator T by

$$\begin{cases} Tf(0) = f(0), \\ Tf(e^{i\theta}) = f(0) - f(e^{i(\theta+\theta_0)}). \end{cases}$$

It is easy to verify that T is an isometry, not of the form (3.10). On the other hand, for an operator of the form (3.10) to be an isometry we must have $C = 1$, and the map φ, rather than satisfy (3.2), need only preserve local distances. To stress this point, we present a second example.

Let $\alpha = 1/2$ and $H = K = [0, 5] \subset R$; d is the natural metric on H and ρ is defined by

$$\rho(x, y) = \begin{cases} |x - y| & \text{if } |x - y| \leqq 4, \\ 2\sqrt{|x - y|} & \text{if } |x - y| > 4. \end{cases}$$

By the remark after Theorem 2.1, only "small" distances need to be considered in determining $\|f\|_\alpha$ for any f (in our case distances no larger than 4), and therefore the identity operator is an isometry. However φ (the identity map in $[0, 5]$) is not globally distance preserving.

Based on these two examples we present the following definitions.

Definition 3.1. We call an operator as in (3.9) an elementary isometry of the first type if and only if there exist

(a) $\theta : K \to \{-1, 1\}$ (we denote θ_u instead of $\theta(u)$),

(b) a bijection $\varphi : K \to H$,

which satisfy, for all $(u, v) \in W_K$,

(3.11.a) $$\rho^\alpha(u, v) < 2 \Leftrightarrow d^\alpha(\varphi(u), \varphi(v)) < 2,$$

(3.11.b) $$\rho^\alpha(u, v) < 2 \Rightarrow \theta_u = \theta_v,$$

(3.11.c) $$\rho^\alpha(u, v) < 2 \Rightarrow \rho(u, v) = d(\varphi(u), \varphi(v)),$$

such that

(3.10′) $$Tf(u) = \theta_u f(\varphi(u)) \qquad \forall f \in \mathrm{lip}^\alpha(H, d), \quad \forall u \in K.$$

Definition 3.2. A metric space (S, Δ) is said to be 1-centered if there exists an $s_0 \in S$ such that $\Delta(s, s_0) = 1$, $\forall s \neq s_0$. In this case we say that s_0 is the center of S and denote $\tilde{S} = S \setminus \{s_0\}$.

Definition 3.3. If (H, d) and (K, ρ) are compact 1-centered metric spaces (with respective centers h_0 and k_0), we call an operator T as in (3.9) an elementary isometry of the second type if and only if there exists a $\theta = \pm 1$ and a distance preserving bijection $\varphi : K \to H$ such that, for all $f \in \mathrm{lip}^\alpha(H, d)$,

$$\begin{cases} Tf(k_0) = \theta f(h_0), \\ Tf(k) = \theta(f(h_0) - f(\varphi(k))), \quad k \neq k_0. \end{cases}$$

Notation. If (S, Δ) is a metric space, $S_1, \cdots, S_n$ are n disjoint 1-centered subspaces and $S_0 = S \setminus \bigcup_{i=1}^n S_i$, we shall write $S = \bigcup_{i=0}^n \oplus S_i$ if and only if $\Delta^\alpha(\tilde{S}_i, S \setminus S_i) \geqq 2$, $i = 1, \cdots, n$. (In other words, if and only if the 1-centered "components" of S are sufficiently isolated among themselves and from the rest of S.)

One can check, generalizing the two given examples, that these two types of "elementary isometries" are in fact isometries. Actually all the isometries are generated in a certain sense by these elementary isometries. More precisely:

Theorem 3.2. *An operator $T : \mathrm{lip}^\alpha(H, d) \to \mathrm{lip}^\alpha(K, \rho)$ is an isometry if and only if:*

(a) *T is an elementary isometry of the first type, or*

(b) *the following conditions are satisfied:*

(i) *there exist $n \geqq 1$, $H_1, \cdots, H_n$ 1-centered subspaces of H and $K_1, \cdots, K_n$*

1-*centered subspaces of* K *such that if* $H_0 = H \setminus \bigcup_{i=1}^n H_i$ *and* $K_0 = K \setminus \bigcup_{i=1}^n K_i$ *then*

$$H = \bigcup_{i=0}^{n} \oplus H_i \quad \textit{and} \quad K = \bigcup_{i=0}^{n} \oplus K_i ;$$

(ii) *there exist* $T_i : \mathrm{lip}^\alpha (H_i, d) \to \mathrm{lip}^\alpha (K_i, \rho)$, $1 \leqq i \leqq n$ *elementary isometries of the second type and* $T_0 : \mathrm{lip}^\alpha (H_0, d) \to \mathrm{lip}^\alpha (K_0, \rho)$ *an elementary isometry of the first type, such that for all* $f \in \mathrm{lip}^\alpha (H, d)$

$$(Tf)\big|_{K_i} = T_i(f\big|_{H_i}), \qquad i = 0, \cdots, n.$$

Proof. It is not difficult to verify that an operator given by (a) or (b) is an isometry. We shall prove the "only if" assertion. We again make use of the fact that

$$T^*(\mathrm{ext}(B_{\mathrm{lip}^\alpha(K,\rho)'})) = \mathrm{ext}(B_{\mathrm{lip}^\alpha(H,d)'}).$$

There are two possibilities:

Case I. $T^*(\{\pm\delta_y / y \in K\}) \subset \{\pm \delta_x / x \in H\}$. In this case we can write

$$T^*\delta_y = \theta_y \delta_{\varphi(y)} \qquad \forall y \in K \tag{3.12}$$

where $\varphi : K \to H$ is a one to one map and $\theta_y = \pm 1$ for all $y \in K$.

We remark that the above inclusion is an equality, since by (3.12) for each pair (y_1, y_2) in K,

$$T\delta_{y_1,y_2} = \rho^{-\alpha}(y_1, y_2)(\theta_{y_1}\delta_{\varphi(y_1)} - \theta_{y_2}\delta_{\varphi(y_2)})$$

and no δ_x can be obtained in this way. Assume now that $y, y' \in K$ and $0 < \rho^\alpha(y, y') < 2$. Then

$$T^*\delta_{y,y'} = \frac{\theta_y\delta_{\varphi(y)} - \theta_{y'}\delta_{\varphi(y')}}{\rho^\alpha(y, y')} \in \exp(B_{\mathrm{lip}^\alpha(H,d)'})$$

whence $\theta_y = \theta_{y'}$ and $\rho(y, y') = d(\varphi(y), \varphi(y'))$. Since $(T^*)^{-1}$ is given by $(T^*)^{-1}\delta_x = \theta_{\varphi^{-1}(x)}\delta_{\varphi^{-1}(x)}$, $d^\alpha(x, x') < 2 \Rightarrow \rho^\alpha(\varphi^{-1}(x), \varphi^{-1}(x')) < 2$. Therefore (3.11) holds and since (3.12) is another way of writing (3.10′), T is an elementary isometry of the first type.

Case II. There exists a $y' \in K$ and $x_0, x_0' \in H$ such that $T^*\delta_{y'} = \delta_{x_0,x_0'}$. Let $T^*\psi = \delta_{x_0}$ ($\psi \in \mathrm{ext}(B_{\mathrm{lip}^\alpha(K,\rho)'})$). We then have $T^*(d^\alpha(x_0, x_0')\delta_{y'} - \psi) = -\delta_{x_0'}$ and therefore

$$\lambda = (d^\alpha(x_0, x_0')\delta_{y'} - \psi) \in \operatorname{ext}(B_{\operatorname{lip}^\alpha(K,\rho)'}).$$

We can thus conclude that either

(i) $\exists y_0 \in K$ such that $\psi = \delta_{y_0}$, or

(ii) $\exists y_0 \in K$ such that $\psi = \delta_{y',y_0}$.

In either case we have

$$d(x_0, x_0') = \rho(y', y_0) = 1. \tag{3.13}$$

We now denote $U = \{y \in K / \rho^\alpha(y, y') < 2\}$ and assume that possibility (i) holds. Let $y \in U$ (not y_0 or y'). Denoting $\mu = T^*\delta_y$ we have

$$T^*\delta_{y,y'} = \frac{\mu - \delta_{x_0,x_0'}}{\rho^\alpha(y, y')} \in \operatorname{ext}(B_{\operatorname{lip}^\alpha(H,d)'}). \tag{3.14}$$

Since μ can't be δ_{x_0} or $-\delta_{x_0}$ there must exist an $x \in H$ such that $\mu = \delta_{x_0,x}$ or $\mu = \delta_{x,x_0}$. If $\mu = \delta_{x,x_0}$ and denoting $\eta = (d^\alpha(x, x_0')\delta_y - \delta_{y',y_0})$, we would have $T^*\varphi = \delta_x$, by which $\eta \in \operatorname{ext}(B_{\operatorname{lip}^\alpha(K,\rho)'})$, an impossibility. Hence $\mu = \delta_{x_0,x}$, $T^*(\delta_{y_0} - d^\alpha(x_0, x)\delta_y) = \delta_x$ and thus $d(x_0, x) = \rho(y_0, y) = 1$. In other words

$$U \setminus \{y_0\} = \{y : \rho(y, y_0) = 1\}. \tag{3.15}$$

Also, from (3.14) we obtain that $d(x, x_0') = \rho(y, y')$. We can therefore define a distance preserving surjection $\varphi : U \setminus \{y_0\} \to \{x : d(x, x_0) = 1\}$ by means of the equation

$$T^*\delta_y = \delta_{x_0,\varphi(y)} \qquad \forall y \in U \setminus \{y_0\}. \tag{3.16}$$

Similarly, if it is possibility (ii) that holds, we also obtain (3.15), and in this case the distance preserving surjection $\varphi : U \setminus \{y_0\} \to \{x : d(x, x_0') = 1\}$ is defined by

$$T^*\delta_y = \delta_{\varphi(y),x_0'} \qquad \forall y \in U \setminus \{y_0\}. \tag{3.16'}$$

Now, for each $y \in K$ we denote $U(y) = \{y' : \rho^\alpha(y, y') < 2\}$. Consider the family $\mathcal{F} = \{U(y) : \exists x \in H \text{ such that } T^*\delta_y = \pm\delta_x\}$. From K's compactness we conclude that there exist $y_1, \cdots, y_m \in K$ such that $K = \bigcup_{i=1}^m U(y_i)$, with $U(y_i) \in \mathcal{F}$ for each i. Without loss of generality we may assume that there exists an integer n, $1 \leqq n \leqq m$, such that for each i not larger than n there exists a $y_i' \in U(y_i)$ such that $T^*\delta_{y_i'}$ is of the form $\delta_{x,x'}$.

Now fix i, $1 \leqq i \leqq n$. Denoting $K_i = U(y_i')$ it follows from (3.15) that K_i is a 1-centered space (y_i being its center); then, define $\theta_i = \pm 1$ and $x_i \in H$ by means of the equation

$$T^*\delta_{y_i} = \theta_i\delta_{x_i} \tag{3.17}$$

and let H_i be the 1-centered metric space given by $H_i = \{x_i\} \cup \{x : d(x, x_i) = 1\}$. It follows from the preceding discussion (see (3.16) and (3.16′)) that there exists a distance preserving surjection $\varphi_i : \tilde{K}_i \to \tilde{H}_i$ such that

$$T^*\delta_y = \theta_i \delta_{x_i, \varphi_i(y_i)} \qquad \forall y \in \tilde{K}_i. \tag{3.18}$$

Clearly K_i and H_i are compact, so if we define

$$T_i : \mathrm{lip}^\alpha (H_i, d) \to \mathrm{lip}^\alpha (K_i, \rho)$$

by

$$\begin{cases} T_i f(y_i) = \theta_i f(x_i), \\ T_i f(y) = \theta_i (f(x_i) - f(\varphi_i(y))), \quad y \in \tilde{K}_i, \end{cases}$$

T_i is an elementary isometry of the seond type. Moreover, from (3.17) and (3.18) it follows that, for all $f \in \mathrm{lip}^\alpha (H, d)$, $(Tf)|_{K_i} = T_i (f|_{H_i})$.

We now consider the (compact) subspaces $H_0 = H \setminus \bigcup_{i=1}^n \tilde{H}_i$ and $K_0 = K \setminus \bigcup_{i=1}^n \tilde{K}_i$. From the definition of H_i, K_i, $i = 1, \cdots, n$, it follows that for each $y \in K_0$ there exists a $\theta_y = \pm 1$ and a bijection $\varphi_0 : K_0 \to H_0$ such that

$$T^*\delta_y = \theta_y \delta_{\varphi_0(y)}. \tag{3.19}$$

Just as in Case I we obtain (3.11) (replacing φ by φ_0). We now define the elementary isometry of the first type $T_0 : \mathrm{lip}^\alpha (H_0, d) \to \mathrm{lip}^\alpha (K_0, \rho)$ by $T_0 f(y) = \theta_y f(\varphi_0(y))$; it follows from (3.19) that $(Tf)|_{K_0} = T_0(f|_{H_0})$ for all $f \in \mathrm{lip}^\alpha (H, d)$.

Finally, the fact that $K = \bigcup_{i=0}^n \oplus K_i$ and $H = \bigcup_{i=0}^n \oplus H_i$ follows basically from (3.15). We leave the details of the verification to the reader. The proof is now complete. □

Before considering the isometries in the remaining cases, we compare Theorem 3.2 with results obtained in [2] and [9]. In [2] H and K are the circle in R^2 with unit circumference. The only isometries obtained were elementary isometries of the first kind since there are no 1-centered subspaces. Moreover, in this case the map φ must assume the form $\theta \to (\theta_0 + \theta)$ or $\theta \to (\theta_0 - \theta)$ (in polar representation) for some θ_0, $0 \leqq \theta_0 < 2\pi$. In general, for an isometry to possess a "second type component", the involved metric spaces must have a very particular structure.

In [9] H and K were assumed to be Riemannian manifolds. Again all isometries were found to be elementary isometries of the first type; furthermore φ turns out to be globally distance preserving (this follows from the fact that the

Riemannian metric is completely determined by small distances, which are preserved by φ).

We now turn to operators of the form

$$T: \mathrm{Lip}^\alpha (H, d; x_0) \to \mathrm{Lip}^\alpha (K, \rho; u_0) \tag{3.20}$$

and

$$T: \mathrm{Lip}^\alpha (H, d) \to \mathrm{Lip}^\alpha (K, \rho). \tag{3.21}$$

We claim that for T as in (3.20) or (3.21) to be an isometry the "same" necessary and sufficient conditions hold as in Theorem 3.1 and Theorem 3.2 respectively. (We assume from now on that the terms $\mathrm{lip}^\alpha(H, d)$, $\mathrm{lip}^\alpha(K, \rho)$, $\mathrm{lip}^\alpha(H, d; x_0)$ and $\mathrm{lip}^\alpha(K, \rho; u_0)$ are respectively replaced by $\mathrm{Lip}^\alpha(H, d)$, $\mathrm{Lip}^\alpha(K, \rho)$, $\mathrm{Lip}^\alpha(H, d; x_0)$ and $\mathrm{Lip}^\alpha(K, \rho; u_0)$ in Theorems 3.1 and 3.2 and in Definitions 3.1 and 3.3.) Indeed, considering Theorem 3.2, the only difference which can arise in its proof is that it is not *a priori* clear that $T^*\delta_y$ belongs to $F(H)$ (see notation in Section 2) for every y. However, this fact is guaranteed by Theorem 2.3: T^*, being a homeomorphism in the respective ω^*-topologies, maps points which possess metrizable neighbourhoods in ω^*-cl $F(K)$ to points with the same property in ω^*-cl $F(H)$. That is to say $T^*(F(K)) \subset F(H)$ (actually equality holds). Once we know that $T^*(F(K)) = F(H)$ the proof is the same as in Theorem 2.3.

The same considerations apply for isometries of the form (3.21). Summing up we obtain

THEOREM 3.3. (I) *An operator as in* (3.20) *is an isometry if and only if there exists a* $\theta = \pm 1$ *and a bijection* $\varphi: K \to H$ *satisfying* $\rho(u, v) = Cd(\varphi(u), \varphi(v))$, $u, v \in K$ (*with* $C = \mathrm{diam}(K)/\mathrm{diam}(H)$) *such that, for all* $g \in \mathrm{Lip}^\alpha(H, d)$ *and all* $u \in H$,

$$Tg(u) = \theta C^\alpha (g(\varphi(u)) - g(\varphi(u_0))).$$

(II) *An operator as in* (3.21) *is an isometry if and only if*

(a) T *is an elementary isometry of the first type, or*

(b) *the following conditions are satisfied*:

(i) *there exist* $n \geqq 1$, $H_1, \cdots, H_n$ 1-*centered subspaces of* H *and* $K_1, \cdots, K_n$ 1-*centered subspaces of* K *such that if* $H_0 = H \setminus \bigcup_{i=1}^n H_i$ *and* $K_0 = K \setminus \bigcup_{i=1}^n K_i$, *then*

$$H = \bigcup_{i=0}^{n} \oplus H_i \quad \textit{and} \quad K = \bigcup_{i=0}^{n} \oplus K_i;$$

(ii) *there exist* $T_i : \mathrm{Lip}^\alpha(H_i, d) \to \mathrm{Lip}^\alpha(K_i, \rho)$, $1 \leqq i \leqq n$ *elementary isometries of the second type and* $T_0 : \mathrm{Lip}^\alpha(H_0, d) \to \mathrm{Lip}^\alpha(K_0, \rho)$ *an elementary isometry of the first type, such that for all* $f \in \mathrm{Lip}^\alpha(H, d)$

$$(Tf)\big|_{K_i} = T_i(f\big|_{H_i}), \qquad i = 0, \cdots, n.$$

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